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TEMPERATURE DISTRIBUTION IN AN IRREGULARITY IN RADIATIVE EQUILIBRIUM

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The problem of radiative equilibrium in a cylinder surrounded by a medium with an arbitrary temperature distribution is analyzed. An analytic solution is obtained for an absorption coefficient depending linearly on optical depth. The solution is applied to the problem of the temperature distribution in a photospheric column located within a sunspot umbra and supplied with energy from below.

1. INTRODUCTION

During the past few years special attention has been devoted to the small bright structures in sunspot umbrae. The physical properties of these formations still are not well established, but their diameter is generally 100-200 km, their brightness is comparable to that of the undisturbed photosphere, and they persist for some tens of minutes [1]. Their magnetic field appears to be considerably weaker than in the umbra itself (according to some estimates the difference may amount to an order of magnitude [1-3]), and its sign is opposite to the sign of the main field of the sunspot.

By taking these irregularities into account one can dispose of various difficulties with discharge models for the sunspot umbra; in particular, one can explain the observed line intensities for neutral and ionized elements, the variation in the average intensity of sunspots from the center to the limb of the solar disk, and the strong departures from hydrostatic equilibrium at the center of a spot [4, 5]. At present the nature of these structures remains unclear. They might be a manifestation of remnants of ordinary convection in the sunspot umbra [6], or of oscillatory convection [7]. Presumably, in regions where the magnetic field is weak, columns of photospheric material would remain unchanged or little changed. On the other hand, it is possible that these bright points might represent isolated formations whose size is limited in depth, as though they were suspended within the umbra [8]. But problems then arise in explaining the heating of the formations [8].

There are also special difficulties with attempting to explain the bright structures as vertical columns that acquire their energy from below. A hot column of considerable vertical length ought to heat the ambient medium by horizontal transfer of radiation, leading to a blurring of its boundaries and an increase in the effective diameter of the cylinder. This effect would be less appreciable if the surrounding umbra material were strongly discharged and if its opacity were low. But another difficulty arises in connection with the discharge of umbra material: While observing at an angle through the rarefied upper layers, one would look through the hot lower layers of the photospheric column. It should be recognized, however, that horizontal energy transfer at the interface between the hot column and the ambient umbra material would tend to produce a boundary layer of diminished temperature. Since the column has a substantially higher opacity than the ambient material, this effect would result in a shielding of the hot inner portion of the column.

In this investigation we have solved the problem of radiative transfer in a cylinder surrounded by a medium whose temperature distribution may be specified arbitrarily. The results will be applied for establishing the temperature distribution in a photospheric column embedded in cool umbra material, assuming that heat transfer takes place by radiation alone.

2. RADIATIVE HEAT TRANSFER IN THE CYLINDER

In a steady state, if local thermodynamic equilibrium prevails, the radiant-flux vector will be solenoidal [9]:

$$\operatorname{div} \mathbf{S} = 0; \tag{1}$$

here $\mathbf{S}$ denotes the flux integrated over all frequencies. In the diffusion approximation we have

$$\mathbf{S} = -\frac{1}{3}\frac{c}{\alpha}\nabla B, \tag{2}$$

where α is the volume absorption coefficient averaged over frequencies and $B = c^{-1}\int I d\Omega$ is the radiation density.

Equations (1) and (2) imply that

$$\nabla\left(\frac{\nabla B}{\alpha}\right) = 0. \tag{3}$$

In a homogeneous medium Eq. (3) reduces to Laplace's equation. We are to solve for Eq. (3) a boundary-value problem in a cylinder of radius R. Let us write Eq. (3) in the cylindrical coordinates r, z:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B}{\partial r}\right) + \frac{\partial^2 B}{\partial z^2} - \frac{\partial B}{\partial r}\frac{\partial \ln \alpha}{\partial r} - \frac{\partial B}{\partial z}\frac{\partial \ln \alpha}{\partial z} = 0. \tag{4}$$

In the dimensionless variables

$$\tau = \int_0^z \alpha(z', 0)\, dz', \qquad \rho = \frac{r}{R},$$

Eq. (4) becomes

$$\frac{\partial^2 B}{\partial \rho^2} - \frac{\partial \ln \alpha}{\partial \rho}\frac{\partial B}{\partial \rho} + \frac{1}{\rho}\frac{\partial B}{\partial \rho} + R^2[\alpha(\tau, 0)]^2 \times\left\{\frac{\partial^2 B}{\partial \tau^2} + \left[\frac{\partial \ln \alpha(\tau, 0)}{\partial \tau} - \frac{\partial \ln \alpha}{\partial \tau}\right]\frac{\partial B}{\partial \tau}\right\} = 0. \tag{5}$$

An application of Eddington's method leads to the same equation (5) (see Buslavskii [10]), which essentially corresponds to the diffusion approximation. At the top and bottom bases of the cylinder we shall prescribe the usual boundary conditions

$$\left(B + \frac{2}{3}\frac{\alpha(\tau, 0)}{\alpha(\tau, \rho)}\frac{\partial B}{\partial \tau}\right)_{\tau=\tau_0} = \Phi, \tag{6}$$

$$\left(B - \frac{2\alpha(\tau, 0)}{3\alpha(\tau, \rho)}\frac{\partial B}{\partial \tau}\right)_{\tau=0} = 0 \tag{7}$$

corresponding to an absence of radiation incident on the cylinder from above, and a certain specified integral flux incident on the cylinder from below.

On the side surface of the cylinder we shall specify the behavior of the function B itself:

$$B|_{\rho=1} = B_1, \tag{8}$$

where $B_1(\tau)$ is some given function.

The variables in Eq. (5) may be separated if we assume that α can be represented in the form

$$\alpha(\tau, \rho) = \alpha(\tau)\beta(\rho). \tag{9}$$

With the substitution $B(\tau, \rho) = T(\tau)Y(\rho)$, Eq. (5) will then reduce to

$$R^2\alpha^2\frac{d^2T}{d\tau^2} + \lambda^2 T = 0, \tag{10}$$

$$\frac{d^2Y}{d\rho^2} + \left(\frac{1}{\rho} - \frac{d\ln\beta}{d\rho}\right)\frac{dY}{d\rho} - \lambda^2 Y = 0. \tag{11}$$

In order that we may integrate Eqs. (10) and (11) in quadratures we should adopt some further simple assumptions regarding the behavior of $\alpha(\tau)$ and $\beta(\rho)$. We shall henceforth suppose that

$$\beta = \text{const}, \tag{9a}$$

$$\alpha = a\tau + \alpha_t. \tag{9b}$$

The linear relation (9b) corresponds approximately to a state of hydrostatic equilibrium.

In addition we shall assume that Φ = const.

Equation (5), with α given by Eq. (9b) and with the boundary conditions (6) and (7), will be satisfied by the linear function

$$B_l = \frac{{}^2/_3 a + \alpha - \alpha_t}{{}^4/_3 a + \alpha_b - \alpha_t}\Phi, \tag{12}$$

where α_b, α_t represent the values of α at the bottom and top bases of the cylinder.

Thus the solution of our original problem reduces to finding a function $u \equiv B - B_l$ that satisfies the same equation (5) with homogeneous boundary conditions.

From Eq. (10), with boundary conditions homogeneous with respect to τ, we find the following eigenvalues of the problem:

$$\lambda_0 = \frac{Ra}{2}\sqrt{1-\mu_0^2}, \quad \lambda_k = \frac{Ra}{2}\sqrt{1+\mu_k^2} \quad (k = 1, 2, \ldots), \tag{13}$$

where the μ_k are defined by the conditions

$$\frac{1}{2}\mu_0 \ln\frac{\alpha_b}{\alpha_t} = \mathrm{Arth}\frac{\mu_0(b+t)}{(b+1)(1-t)-\mu_0^2}, \quad (14a)$$

$$\frac{1}{2}\mu_k \ln\frac{\alpha_b}{\alpha_t} = \mathrm{arc\,tg}\frac{\mu_k(b+t)}{(b+1)(1-t)+\mu_k^2} + k\pi \quad (14b)$$

$$(k = 1, 2, \ldots).$$

Here $b = 3\alpha_b/a$ and $t = 3\alpha_t/a$. It is natural to suppose that $\alpha_t/a \ll 1$ (the mass density in the upper layers of the cylinder is low), so that $t \ll 1$ and $b \approx 3\tau_0$. Equation (14a) then implies that $\mu_0 \approx 1$ to within terms of order α_t/a; to the same accuracy $\lambda_0 \approx 0$, and Eq. (14b) becomes

$$\frac{1}{2}\mu_k \ln\frac{\alpha_b}{\alpha_t} \approx \mathrm{arc\,tg}\frac{3\mu_k\tau_0}{1+3\tau_0+\mu_k^2} + k\pi. \quad (14c)$$

The eigenfunctions have the form

$$T_0 = 1 - \frac{\alpha}{\alpha_b}; \quad (15a)$$

$$T_k = \left(\frac{\alpha}{\alpha_b}\right)^{1/2} \sin\left(\frac{1}{2}\mu_k \ln\frac{\alpha_b}{\alpha} + \varphi_k\right), \qquad k = 1, 2, \ldots, \quad (15b)$$

where

$$\varphi_k = \mathrm{arctg}\frac{\mu_k}{b+1}. \quad (16)$$

The Bessel functions of imaginary argument,

$$Y_n = I_0(\lambda_n \rho), \quad n = 0, 1, \ldots \quad , \quad (17)$$

will be solutions of Eq. (11) for $\beta = \mathrm{const}$, satisfying the condition of boundedness along the axis. Thus the solution of the problem will take the form

$$B = B_l + C_0\left(1 - \frac{\alpha}{\alpha_b}\right) I_0(\lambda_0\rho) + \sum_{k=1} C_k T_k(\alpha) I_0(\lambda_k \rho). \quad (18)$$

The coefficients C_0, C_k may be evaluated by expanding the function $u_1(\alpha) \equiv B_1(\alpha) - B_l(\alpha)$ in series with respect to the eigenfunctions $T_n(\alpha)$. Assuming for simplicity that $u_1(\alpha)$ is linear,

$$u_1(\alpha) = (f\alpha + g)B_\odot ,$$

we find, neglecting terms of order α_t/a, that

$$C_0 = gB_\odot, \qquad C_k = 4B_\odot\left[\left(\alpha_b + \frac{2}{3}a\right)f + g\right]\frac{\mu_k}{\sqrt{1+\mu_k^2}}$$

$$\times \frac{1}{I_0(\lambda_k)} \frac{\sin\psi_k}{\gamma_k + \sin(2\varphi_k) - \sin(\gamma_k + 2\varphi_k)}, \quad (19)$$

where the φ_k are given by Eq. (16),

$$\gamma_k = \mu_k \ln\frac{\alpha_b}{\alpha_t}, \quad (20)$$

$$\psi_k = \mathrm{arc\,tg}\frac{b\mu_k}{b+1+\mu_k^2}, \quad (21)$$

and $B_\odot$ denotes the density of radiation emerging from the undisturbed photosphere.

3. PHOTOSPHERIC COLUMN IN SUNSPOT UMBRA

We shall now apply the solution obtained in Sec. 2 to the case of a photospheric column surrounded by the cool, rarefied material of a sunspot umbra. For this purpose we shall take Φ to be the flux in the undisturbed photosphere at the same (sufficiently great) optical depth τ_0; that is,

$$\Phi = B_\odot\left(2 + \frac{3}{2}\tau_0\right). \quad (22)$$

Such a flux would, in the one-dimensional case, give a radiation density $B_\odot$ at the surface. With this value of Φ the linear term will become

$$B_l = \left(1 + \frac{3}{2}\tau\right)B_\odot. \quad (12a)$$

Figure 1 illustrates the form of the isotherms inside the photospheric column for the following values of the parameters:

$$R = 100 \text{ km}, \quad a = 1.5 \cdot 10^{-2} \text{ km}^{-1}, \quad \frac{\alpha_b}{\alpha_t} = 10^5,$$

$$\tau_0 = 10, \quad f = -100 \text{ km}, \quad g = -0.5.$$

This choice of f and g ensures constancy of temperature on the side surface [$B_1 = (1+g)B_\odot$]. The values of the other parameters are approximately the same as those in Michard's model photosphere [11].

Knowing the temperature distribution inside the column, we can estimate the brightness of the column in various directions. Thus, the radiant intensity on axis in the direction of the z axis will be

$$I_0 = \int_0^\infty Be^{-\tau}\,d\tau.$$

A calculation shows that if $g = 0.5$, then I_0 will constitute about one half the radiant intensity of the undisturbed photosphere, in rough agreement with the results obtained by Beckers and Schröter [1]; with this value of g the radiant intensity of the umbra will amount to 20% of the photospheric value.

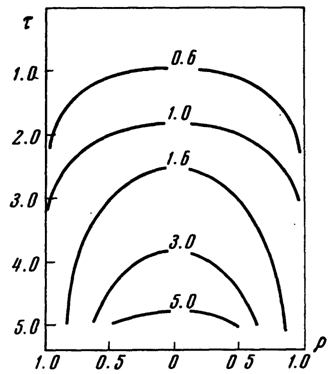

Fig. 1. Form of isotherms inside the photospheric column as a function of the optical depth τ and the dimensionless distance ρ from the axis of the cylinder [see Eq. (5)]. The curves are labeled with the value of $B/B_{\odot}$.

If the irregularity is observed at an angle to its axis, its brightness will be considerably lower than would be the case for a pure photospheric column because the temperature near the side of the cylinder is substantially lower than on the axis.

Thus our model for a column of photospheric material with energy supplied from below qualitatively represents the properties now known for the bright formations in sunspot umbrae. But one important question remains: To what extent is the model itself consistent? Its weakest point is the specification of the temperature on the side surface. Instead of calculating the heating of the rarefied material by the radiation of the hot column, we have replaced that material by a thermostat in which the column is embedded. Our results partially justify this procedure: The temperature falls off fairly rapidly in the dense boundary layers of the column, while the surrounding rarefied material is only weakly heated. Nevertheless, there is an urgent need to solve the combined radiative-transfer problem in a spot where the irregularities are taken into account.

The other assumptions we have made in solving the problem are not of such a fundamental character and would not affect the result qualitatively. The value $\tau_0 = 10$ that we have adopted is of no essential importance. Generally speaking, the solution does not depend on the value of τ_0 throughout the layer $\tau < \tau_0/2$. Nor is the form of the boundary condition (6) critical: In the region of physical interest, $\tau < 5$, the same solution is obtained under the more general condition

$$\left(B + \frac{2}{3} k \frac{\partial B}{\partial \tau}\right)_{\tau=\tau_0} = \Phi \qquad (0 \leqslant k \leqslant 1). \qquad \text{(6a)}$$

In solving the problem we have assumed that the absorption coefficient is a particular function of depth. The transfer problem essentially ought to be solved in conjunction with the equations of magnetohydrodynamics. The results obtained here ought to be viewed as a first approximation. The behavior of the isotherms could be determined more accurately by taking the temperature dependence of α into account. Qualitatively it is clear that a decrease of α in the surface layers would result in some brightening of the column.

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