SCATTERING MATRIX FOR RADIATION IN A MAGNETIC FIELD

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The resonance scattering matrix for radiation in a magnetic field is derived. Agreement has been achieved with the experiments of Hanle and the formula of Van Vleck.

There has been increased interest in recent years in astrophysics in the theory of formation of spectral lines in stellar atmospheres in the presence of magnetic fields. This is due to the fact that the presence of such fields, even if their intensity is quite low, may have an appreciable effect both on the magnitude and direction of the polarization of the escaping radiation. Therefore, any studies of the polarization of radiation, particularly at the solar limb, in prominences, and so on, must take into account the possible influence of the magnetic field. It has been shown [1, 2] that the physical parameters of sunspots and of the atmospheres of magnetic stars, determined by the curve-of-growth method, are modified when the magnetic field is taken into account. Finally, the theory of formation of spectral lines in the atmospheres of magnetic stars is of particular importance in connection with the development of magnetographs for the determination of the three components of the solar magnetic fields [3-4]. Adequate correlation between signals measured directly by the instruments and the magnitude and direction of the magnetic field can only be obtained if a correct theory of line formation is available.

The equation for the transport of polarized radiation in a nonisotropic medium as formulated by Rozenberg [5] may be written in the form

$$\mu \frac{dS_i}{d\tau} = (1 + \eta_{ik}) S_k - (1 - \varepsilon) \times \int D_{ik}(\mu, \mu') S_k(\mu') \frac{d\omega'}{4\pi} - (1 + \varepsilon\eta_{ik}) B_\nu. \quad (1)$$

Here the vector S_i is set up from the Stokes parameters which we shall define as follows. Let the electric vector of arbitrarily polarized light in the system of coordinates which we have selected be written in the form

$$x = \xi_1 \cos(\omega t - \varepsilon_1), \; y = \xi_2 \cos(\omega t - \varepsilon_2). \quad (2)$$

The Stokes parameters will then be defined by

$$S_1 = I = \xi_1^2 + \xi_2^2, \quad S_2 = Q = \xi_1^2 - \xi_2^2, \quad S_3 = U = \overline{2\xi_1\xi_2 \cos(\varepsilon_1 - \varepsilon_2)}, \quad S_4 = V = \overline{2\xi_1\xi_2 \sin(\varepsilon_1 - \varepsilon_2)}. \quad (3)$$

There are other ways of defining these parameters. The general form of Eq. (1) will then remain the same but the special forms of the absorption matrix η_{ik} and scattering matrix D_{ik} will change.

The absorption matrix η_{ik} for the above Stokes parameters was derived by Unno [6]. Stepanov [7, 8] derived the absorption matrix independently for the generalized Stokes parameters $I_\pm$, for which the matrix η_{ik} becomes diagonal. Rachkovskii [9, 10] has shown that the absorption matrices obtained by Unno and Stepanov are equivalent if one neglects effects associated with anomalous dispersion, and has established a more general expression in which these effects are allowed for.

The situation is somewhat more complicated in the case of the scattering matrix D_{ik}. Unno does not take scattering into account, while Stepanov [7] solves the transport equation on the assumption that the absorption and scattering matrices are identical, which is a very rough approxima-

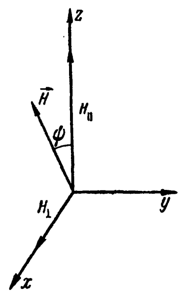

Fig. 1

tion. Stepanov [11] and Rachkovskii [12] subsequently derived more exact expressions for the scattering matrices for a number of special transitions in the system of parameters which they adopted. However, the parameters $I_{\pm}$ do not introduce the required simplification since the matrix D_{ik} does not become diagonal simultaneously with η_{ik}. Below, we shall derive the scattering matrix in a magnetic field for the usual Stokes parameters, using the method employed by Stepanov in [11].

As an example, let us consider the derivation of the scattering matrix for the transition with j = 1 for the upper level, and j = 0 for the lower level. Examples of such lines are the resonance line of mercury λ 2537 A and the well-known magnetic line of iron λ 5250.22 A.

Suppose that arbitrarily polarized radiation is incident on an ensemble of such atoms in a magnetic field. We shall take the axes of the coordinate system as shown in Fig. 1, where the z axis lies along the direction of propagation and the x axis lies along the projection of the magnetic field onto the plane of observation. Absorption at each level can then be written [6] in the form

$$\Delta J_l = k_l\left(I\,\frac{1+\cos^2\psi'}{4} - Q\,\frac{\sin^2\psi'}{4} - V\cdot\frac{\cos\psi'}{2}\right),$$

$$\Delta J_p = k_p\left(I\,\frac{\sin^2\psi'}{2} + Q\cdot\frac{\sin^2\psi'}{2}\right),$$

$$\Delta J_r = k_r\left(I\,\frac{1+\cos^2\psi'}{4} - Q\cdot\frac{\sin^2\psi'}{4} + V\cdot\frac{\cos\psi'}{2}\right). \quad (4)$$

Radiation reaching an atom from any particular direction characterized by the angle ψ' is scattered in all directions. For the scattered radiation we shall select a set of coordinates as above, which will not be linked with the new direction of propagation. The x and y axes, which are linked with the projection onto the plane of observation, will then occupy new positions.

In this system of coordinates the radiation from each level can be written in the form

$$\begin{array}{l|l|l} \qquad\sigma_l & \qquad\pi & \qquad\sigma_r \\ \xi_x = \xi_0\cos\psi\cos\omega t & \zeta_x = \zeta_0\sin\psi\cos\omega t & \eta_x = \eta_0\cos\psi\cos\omega t \\ \xi_y = \xi_0\sin\omega t & \zeta_y = 0 & \eta_y = -\eta_0\sin\omega t \end{array} \quad (5)$$

Hence, the Stokes parameters for radiation from each of the sublevels are

$$\begin{array}{l|l|l} I_l = \xi_0^2(1+\cos^2\psi) & I_p = \zeta_0^2\sin^2\psi & I_r = \eta_0^2(1+\cos^2\psi) \\ Q_l = -\xi_0^2\sin^2\psi & Q_p = \zeta_0^2\sin^2\psi & Q_r = -\eta_0^2\sin^2\psi \\ U_l = 0 & U_p = 0 & U_r = 0 \\ V_l = 2\xi_0^2\cos\psi & V_p = 0 & V = -2\eta_0^2\cos\psi \end{array}$$

We can combine these parameters if we assume that the radiation is emitted independently by the atoms in the ensemble:

$$\begin{aligned} I &= (\xi_0^2+\eta_0^2)(1+\cos^2\psi) + \zeta_0^2\sin^2\psi, \\ Q &= -(\xi_0^2+\eta_0^2)\sin^2\psi + \zeta_0^2\sin^2\psi, \\ U &= 0, \\ V &= 2(-\xi_0^2+\eta_0^2)\cos\psi. \end{aligned} \quad (6)$$

For the transition j = 0, Δj = 1, we have

$$\xi_0^2 = {}^1\!/_2 A_l k_l, \quad \zeta_0^2 = A_p k_p, \quad \eta_0^2 = {}^1\!/_2 A_r k_r. \quad (7)$$

The normalizing factors A_l, A_p, and A_r can be found from the law of conservation of energy for each level:

$$\int_{4\pi} I_l\,d\omega = \Delta J_l, \quad \int_{4\pi} I_p\,d\omega = \Delta J_p, \quad \int_{4\pi} I_r\,d\omega = \Delta J_r. \quad (8)$$

It is evident that

$$\int\frac{1+\cos^2\psi}{2}\,d\omega = \int\sin^2\psi\,d\omega = \frac{8\pi}{3}, \quad (9)$$

in which case

$$A_l = \frac{3}{2}\cdot\frac{1}{4\pi}\left[I\cdot\frac{1+\cos^2\psi'}{4} - Q\cdot\frac{\sin^2\psi'}{4} - V\cdot\frac{\cos\psi'}{2}\right], \quad (10)$$

$$A_p = \frac{3}{2}\cdot\frac{1}{4\pi}\left[I\cdot\frac{\sin^2\psi'}{2} + Q\cdot\frac{\sin^2\psi'}{2}\right], \quad (11)$$

$$A_r = \frac{3}{2}\cdot\frac{1}{4\pi}\left[I\cdot\frac{1+\cos^2\psi'}{4} - Q\cdot\frac{\sin^2\psi'}{4} + V\cdot\frac{\cos\psi'}{2}\right]. \quad (12)$$

Using Eqs. (6), (7), (10), and (12), we can write down at once the components of the scattering matrix:

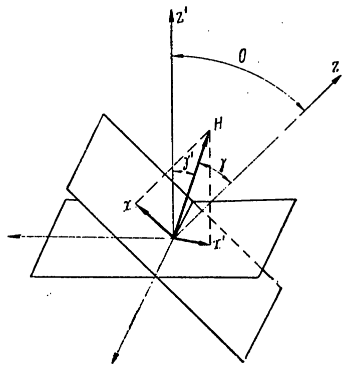

Fig. 2. In the text the angles ψ and ψ' correspond to γ and γ'.

$$
\left.
\begin{aligned}
D_{11} &= \frac{3}{2}\left[\frac{k_l+k_r}{4}\cdot\frac{(1+\cos^2\psi)(1+\cos^2\psi')}{2} + \frac{k_p}{2}\sin^2\psi\sin^2\psi'\right],\\
D_{12} &= \frac{3}{2}\left[-\frac{k_l+k_r}{4}\cdot\frac{1+\cos^2\psi}{2}\sin^2\psi' + \frac{k_p}{2}\sin^2\psi\sin^2\psi'\right],\\
D_{14} &= \frac{3}{2}\cdot\frac{k_r-k_l}{2}\cdot\frac{1+\cos^2\psi}{2}\cos\psi',\\
D_{21} &= \frac{3}{2}\left[-\frac{k_l+k_r}{4}\cdot\frac{1+\cos^2\psi'}{2}\sin^2\psi + \frac{k_p}{2}\sin^2\psi\sin^2\psi'\right],\\
D_{22} &= \frac{3}{2}\left[\frac{k_l+k_r}{4}\cdot\frac{\sin^2\psi'}{2}\sin^2\psi + \frac{k_p}{2}\sin^2\psi\sin^2\psi'\right]\\
D_{24} &= -\frac{3}{2}\left[\frac{k_r-k_l}{2}\cdot\frac{\sin^2\psi}{2}\cos\psi'\right],\\
D_{41} &= \frac{3}{2}\cdot\frac{k_r-k_l}{4}(1+\cos^2\psi')\cos\psi,\\
D_{42} &= -\frac{3}{2}\cdot\frac{k_r-k_l}{4}\sin^2\psi'\cos\psi,\\
D_{44} &= \frac{3}{2}\cdot\frac{k_r+k_l}{2}\cos\psi'\cos\psi,\\
D_{13} &= D_{31} = D_{23} = D_{32} = D_{33} = D_{34} = D_{43} = 0.
\end{aligned}
\right\} \quad (13)
$$

The scattering matrices for other transitions can be established in a similar way. It may be shown that the scattering matrix for the transition $j = 1$, $\Delta j = -1$, which is not reproduced here, is equivalent to the scattering matrix used by Rachkovskii [12].

In comparisons with experiment [14-16] it must be remembered that the matrix (13) connects the Stokes parameters for the absorbed and scattering radiation in different coordinate systems, which is connected with the changed position of the plane of observation after scattering (Fig. 2). If we let χ represent the angle between the line of intersection of the planes of observation and the projection of the magnetic field, we can relate ψ and χ to ψ' and χ' with the aid of the scattering angle θ:

$$\cos\psi = \cos\psi'\cos\theta - \sin\psi'\sin\chi'\sin\theta, \quad (14)$$

$$\tan\chi = \frac{\sin\psi'\sin\chi'\cos\theta + \cos\psi\sin\theta}{\sin\psi'\cos\chi'}. \quad (15)$$

The usual rotation transformation can be written in the form

$$L(\chi-\chi') = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2(\chi-\chi') & \sin 2(\chi-\chi') & 0 \\ 0 & -\sin 2(\chi-\chi') & \cos 2(\chi-\chi') & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}. \quad (16)$$

The magnitude and direction of the polarization obtained from (13) and (16) are in good agreement with Hanle's measurements on the 2537 A polarized resonance line of mercury in an external magnetic field [14, 15, 16].

The polarization predicted by (13) can also be compared with Van Vleck's formula [16], which has been verified experimentally. According to this formula, the magnitude of the polarization for the transition $j = 0$, $\Delta j = 1$ in the case of excitation by linearly polarized light and observation at right-angles to the magnetic field is given by

$$P = \frac{3\cos^2\theta - 1}{1+\cos^2\theta}, \quad (17)$$

where θ is the angle between the electric vector in the exciting light and the magnetic field. If we denote the angle between the electric vector E and the projection of the magnetic field H onto the plane of observation by Φ, then

$$\sin\psi' = \frac{\cos\theta}{\cos\Phi}, \quad (18)$$

$$Q = I\cos 2\Phi. \quad (19)$$

Substituting (18) and (19) into (13) and assuming that the field is small ($k_l = k_p = k_r = k$), we obtain Van Vleck's formula (17) for $\psi = \pi/2$.

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